A 2-ADIC CONTROL THEOREM FOR MODULAR CURVES

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Abstract. We study the behaviour of ordinary parts of the homology modules of modular curves, associated to a decreasing sequence of congruence subgroups $\Gamma_1(N2^r)$ for $r \geq 2$, and prove a control theorem for these homology modules.

1. Introduction

Hida theory studies the modular curves associated to the following congruence subgroups, for primes $p \geq 5$ and $(p, N) = 1$,

$$\cdots \subset \Gamma_1(Np^r) \subset \cdots \subset \Gamma_1(Np). \tag{$*$}$$

Let Y_r denote the Riemann surface associated to the congruence subgroup $\Gamma_1(Np^r)$. One of the important results in Hida theory [3] is that the projective limit of ordinary parts of the homology modules, i.e., $W^{\mathrm{ord}} := \varprojlim_r \mathrm{H}_1(Y_r, \mathbb{Z}_p)^{\mathrm{ord}}$, is a free Λ-module of finite rank and

$$W^{\mathrm{ord}}/\mathfrak{a}_r W^{\mathrm{ord}} = \mathrm{H}_1(Y_r, \mathbb{Z}_p)^{\mathrm{ord}}, \tag{$**$}$$

for all $r \geq 1$, where $\mathfrak{a}_r$ denotes the augmentation ideal of $\mathbb{Z}[[1 + p^r\mathbb{Z}_p]]$ and $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$. In [1], Emerton gave a proof of these results above for primes $p \geq 5$, using algebraic topology of the Riemann surfaces Y_r.

Emerton's proof for $p \geq 5$ holds for $p = 3$ with $N > 1$ verbatim, but for $p = 2$ we show that similar results hold only after passing to smaller congruence subgroups. Moreover, there is no restriction on N, i.e., N can be equal to 1 (unlike when $p = 3$) (cf. Theorem 5.2 in the text). As a consequence of these results, we proved control theorems for ordinary 2-adic families of modular forms, see [2]. Some amount of calculations will be omitted and the reader should refer to those in [1] for more details.

2. Preliminaries

Throughout this note, let $p = 2$, $q = 4$, and $N \in \mathbb{N}$ such that $(p, N) = 1$. We look at the modular curves associated to the following congruence subgroups

$$\cdots \subset \Gamma_1(Np^r) \subset \cdots \subset \Gamma_1(Nq).$$

If we take the homology with $\mathbb{Z}$-coefficients of the tower of modular curves, we get a tower of finitely generated free abelian groups

$$\cdots \to \Gamma_1(Np^r)^{\mathrm{ab}} \to \cdots \to \Gamma_1(Nq)^{\mathrm{ab}}, \tag{2.1}$$

because for $r \geq 2$, $\mathrm{H}_1(\Gamma_1(Np^r)\backslash\mathbb{H}, \mathbb{Z}) = \Gamma_1(Np^r)^{\mathrm{ab}}$, where $\mathbb{H}$ denotes the upper half-plane. To understand (2.1), we introduce the congruence subgroups for $r \geq 2$:

$$\Phi_r^2 = \Gamma_1(Nq) \cap \Gamma_0(p^r).$$

Clearly, we have $\Gamma_1(Np^r) \subset \Phi_r^2 \subset \Gamma_1(Nq)$ and $\Gamma_1(Np^r)$ is a normal subgroup of Φ_r^2. For $r \geq 2$, we define $\Gamma_r := \mathrm{Ker}\ (\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}_p/p^r\mathbb{Z}_p)^\times)$, which is a subgroup of Γ_2 with index p^{r-2}. Set $\Gamma := \Gamma_2$.

We define a morphism of groups

$$\Phi_r^2 \xrightarrow{\eta_r} \Gamma/\Gamma_r$$

via the formula

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longrightarrow d \bmod \Gamma_r. \tag{2.2}$$

Lemma 2.1. *The map η_r is surjective.*

Proof. Given a $\bar{d} \in \Gamma/\Gamma_r$, we can take a lift d of $\bar{d}$ of the form $1 + kqN$ for some $k \in \mathbb{Z}$, because for any $\alpha, \beta \in \Gamma, \alpha \equiv \beta \pmod{\Gamma_r}$ if and only if $\alpha - \beta \in p^r\mathbb{Z}_p$. Now, take c to be Np^r. Clearly $(c, d) = 1$, and hence there exists $a, b \in \mathbb{Z}$ such that $ad - bc = 1$. We see that $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Phi_r^2$ and $\eta_r(\alpha) = \bar{d}$. □

Remark 2.2. *The restriction of η_r to $\Phi_r^2 \cap \Gamma^0(p)$, which we denote by Res(η_r), is also surjective onto Γ/Γ_r. Moreover, we have the following commutative diagram*

$$\begin{tikzcd} \Phi_{r+1}^2 \arrow[r, two heads, "\eta_{r+1}"] \arrow[d, "t^{-1}-t", "\wr"'] & \Gamma/\Gamma_{r+1} \arrow[d] \\ \Phi_r^2 \cap \Gamma^0(p) \arrow[r, two heads, "Res(\eta_r)"] & \Gamma/\Gamma_r, \end{tikzcd}$$

where the group $\Gamma^0(p) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod 2\}$ and $t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$.

By Lemma 2.1, we have the following short exact sequence of groups

$$1 \to \Gamma_1(Np^r) \to \Phi_r^2 \xrightarrow{\eta_r} \Gamma/\Gamma_r \to 1.$$

The action of Φ_r^2 on $\Gamma_1(Np^r)$ by conjugation induces an action of $\Phi_r^2/\Gamma_1(Np^r) = \Gamma/\Gamma_r$ on $\Gamma_1(Np^r)^{\mathrm{ab}}$. Thus Γ acts naturally on $\Gamma_1(Np^r)^{\mathrm{ab}}$. The morphisms in the chain

$$\cdots \to \Gamma_1(Np^r)^{\mathrm{ab}} \to \cdots \to \Gamma_1(Nq)^{\mathrm{ab}}$$

are clearly Γ-equivariant.

If $r \geq s > 1$, we denote by Φ_r^s the subgroup of Φ_r^2 containing $\Gamma_1(Np^r)$ whose quotient by $\Gamma_1(Np^r)$ equals Γ_s/Γ_r, i.e., $\Phi_r^s := \Gamma_1(Np^s) \cap \Gamma_0(p^r)$. Moreover, we have

$$\Gamma_1(Np^r)^{\mathrm{ab}} \to \Phi_r^{s\,\mathrm{ab}} \to \Gamma_s/\Gamma_r \to 1.$$

For any $s > 1$, let γ_s denote a topological generator of Γ_s. Then the augmentation ideal $\mathfrak{a}_s$ of $\Lambda = \mathbb{Z}_p[[\Gamma]]$ is a principal ideal generated by $\gamma_s - 1$. Similarly, for $i > 0$, $\Gamma_{s+i} = \overline{\langle \gamma_s^{p^i} \rangle}$ and $\mathfrak{a}_{s+i} = (\gamma_s^{p^i} - 1)$. Clearly, for any $r \geq s > 1$, the augmentation ideal of $\mathbb{Z}[\Gamma_s/\Gamma_r]$ is $\mathfrak{a}_s$, and

$$\mathfrak{a}_s\Gamma_1(Np^r)^{\mathrm{ab}} = [\Phi_r^s, \Gamma_1(Np^r)]/[\Gamma_1(Np^r), \Gamma_1(Np^r)] \subset \Gamma_1(Np^r)^{\mathrm{ab}},$$

and the last inclusion follows since $\Gamma_1(Np^r)$ is a normal subgroup of Φ_r^s. The extension

$$1 \to \Gamma_1(Np^r)/[\Phi_r^s, \Gamma_1(Np^r)] \to \Phi_r^s/[\Phi_r^s, \Gamma_1(Np^r)] \to \Gamma_s/\Gamma_r \to 1$$

is a central extension of a cyclic group, thus the middle group is abelian, implying that

$$[\Phi_r^s, \Phi_r^s] = [\Phi_r^s, \Gamma_1(Np^r)].$$

The equality holds because of $\Phi_r^s \supseteq \Gamma_1(Np^r)$ and the fact that the commutator subgroup of the group $\Phi_r^s/[\Phi_r^s, \Gamma_1(Np^r)]$ is trivial.

Remark 2.3. *The following diagram is commutative*

$$\begin{tikzcd} \frac{\Phi_r^s \cap \Gamma^0(p)}{\Gamma_1(Np^r)\cap\Gamma^0(p)} \arrow[r, "i"] \arrow[dr, "\sim"] & \frac{\Phi_r^s}{\Gamma_1(Np^r)} \arrow[d, "\sim"] \\ & \frac{\Gamma_s}{\Gamma_r}. \end{tikzcd}$$

The diagonal map is an isomorphism, by Remark 2.2. Since Γ_s/Γ_r is finite, we see that the inclusion i is an isomorphism. This remark is useful in proving Lemma 3.6.

To prove Theorems 4.1 and 5.2, we need to understand the images of these morphisms

$$\Gamma_1(Np^r)^{\mathrm{ab}} \to \Gamma_1(Np^s)^{\mathrm{ab}}$$

in the chain of homology groups as in (2.1). Unfortunately, we do not have a good characterization these images for $r \geq s > 1$ in general, and so we cannot get a good description of the projective limit. This morphism can be factored as

$$\Gamma_1(Np^r)^{\mathrm{ab}} \twoheadrightarrow \Gamma_1(Np^r)^{\mathrm{ab}}/\mathfrak{a}_s \hookrightarrow \Phi_r^{s\,\mathrm{ab}} \longrightarrow \Gamma_1(Np^s)^{\mathrm{ab}},$$

and the problem is that the second and third morphisms may not be isomorphisms, in general.

Hida observed that if one applies a certain projection operator arising from the Atkin U-operator to all these modules then they become isomorphisms, in which case we have a good control over the images of the morphisms in (2.1). So we now define the Atkin U-operator and study their properties.

3. Hecke operators

Suppose G, H are two subgroups of a group T, and $t \in T$ such that $[G : t^{-1}Ht \cap G] < \infty$. Then one has

$$G^{\mathrm{ab}} \xrightarrow{V} (t^{-1}Ht \cap G)^{\mathrm{ab}} \xrightarrow{\sim} (H \cap tGt^{-1})^{\mathrm{ab}} \longrightarrow H^{\mathrm{ab}},$$

where V is the transfer map, the isomorphism is given by conjugating with t, and the last morphism is induced by $H \cap tGt^{-1} \hookrightarrow H$. Taking the composition of all these we obtain a morphism

$$[t] : G^{\mathrm{ab}} \to H^{\mathrm{ab}},$$

the "Hecke operator" corresponding to t.

In our case, take $T = \mathrm{GL}_2(\mathbb{Q})$, $G = H =$ a congruence subgroup of SL_2 of level divisible by p, and $t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. We denote the corresponding Hecke operator by U_2. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Phi_r^s$, we see that

$$t^{-1}At = \begin{pmatrix} a & bp \\ c/p & d \end{pmatrix} \quad \text{and} \quad tAt^{-1} = \begin{pmatrix} a & b/p \\ cp & d \end{pmatrix}.$$

Remark 3.1. *Observe that $(1,1)$, $(2,2)$-entries of A and of $t^{\pm 1}At^{\mp 1}$ are the same.*

It is easy to see that $t^{-1}\Phi_r^s t \cap \Phi_r^s = \Phi_r^s \cap \Gamma^0(p), \Phi_r^s \cap t\Phi_r^s t^{-1} = \Phi_{r+1}^s$, where the group $\Gamma^0(p)$ is as in Remark 2.2. Thus, the Atkin U-operator (resp. U'-operator) is by definition the composition

$$\Phi_r^{s\,\mathrm{ab}} \xrightarrow{V} (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \xrightarrow[\sim]{t-t^{-1}} \Phi_{r+1}^{s\,\mathrm{ab}} \longrightarrow \Phi_r^{s\,\mathrm{ab}}, \tag{3.1}$$

(resp., the composition of the first two of above morphisms).

Lemma 3.2. *Suppose that $r \geq s > 1, r' \geq s' > 1, r \geq r', s \geq s'$, so that $\Phi_r^s \subset \Phi_{r'}^{s'}$. Then the following diagram commutes*

$$\begin{tikzcd}
\Phi_r^{s\,\mathrm{ab}} \arrow[r] \arrow[d, "U'"] & \Phi_{r'}^{s'\,\mathrm{ab}} \arrow[d, "U'"] \\
\Phi_{r+1}^{s\,\mathrm{ab}} \arrow[r] & \Phi_{r'+1}^{s'\,\mathrm{ab}}.
\end{tikzcd}$$

Thus, the Atkin U-operator commutes with the morphism $\Phi_r^{s\,\mathrm{ab}} \to \Phi_{r'}^{s'\,\mathrm{ab}}$.

Proof. The proof is similar to the proof of [1, Lem. 3.1]. The final statement follows from (3.1), since the Atkin U-operator, by definition, is the composition of U'-operator and the morphism induced by the inclusion of groups $\Phi_{r+1}^s \subset \Phi_r^s$. □

Corollary 3.3. *For $r \geq s > 1$, each $\Phi_r^{s\,ab}$ is a $\mathbb{Z}[U]$-module via the action of U and morphisms between these modules (arising from the inclusions) are morphisms of $\mathbb{Z}[U]$-modules. Hence, the cokernels of these morphisms acquire a $\mathbb{Z}[U]$-module structure.*

Suppose π denote the morphism $\pi : \Phi_r^{s\,\mathrm{ab}} \longrightarrow \Phi_{r-1}^{s\,\mathrm{ab}}$ and π' for the morphism $\pi' : \Phi_{r+1}^{s\,\mathrm{ab}} \longrightarrow \Phi_r^{s\,\mathrm{ab}}$. Then, by Lemma 3.2, we have

$$U' \circ \pi = \pi' \circ U' = U \in \mathrm{End}_{\mathbb{Z}}(\Phi_r^{s\,\mathrm{ab}}). \tag{3.2}$$

By the definition of U', we see that $\pi \circ U' = U \in \mathrm{End}_{\mathbb{Z}}(\Phi_{r-1}^{s\,\mathrm{ab}})$.

By Corollary 3.3, the cokernel of the morphism $\Gamma_1(Np^r)^{\mathrm{ab}} \to \Phi_r^{s\,\mathrm{ab}}$, for $r \geq s > 1$, is a $\mathbb{Z}[U]$-module and this cokernel is isomorphic to the group Γ_s/Γ_r. Hence, the group Γ_s/Γ_r is a $\mathbb{Z}[U]$-module. Observe that $\Phi_r^r = \Gamma_1(Np^r)$.

Lemma 3.4. *The operator U acts on Γ_s/Γ_r as multiplication by p.*

Proof. The operator U acts on Γ_s/Γ_r as a multiplication by p if and only if it acts on $\frac{\Phi_r^{s\,\mathrm{ab}}}{\Gamma_1(Np^r)^{\mathrm{ab}}}$ as $\bar{A} \mapsto \bar{A}^p$. The operator U is the composition of the following morphisms:

$$\begin{array}{ccccccc}
\frac{\Phi_r^{s\,\mathrm{ab}}}{\Gamma_1(Np^r)^{\mathrm{ab}}} & \xrightarrow{V} & \frac{(\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}}}{(\Gamma_1(Np^r) \cap \Gamma^0(p))^{\mathrm{ab}}} & \xrightarrow{t-t^{-1}} & \frac{\Phi_{r+1}^{s\,\mathrm{ab}}}{\Phi_{r+1}^{r\,\mathrm{ab}}} & \longrightarrow & \frac{\Phi_r^{s\,\mathrm{ab}}}{\Gamma_1(Np^r)^{\mathrm{ab}}} \\
\bar{A} & \longmapsto & \bar{A}^p & \longmapsto & t\bar{A}^p t^{-1} & \longmapsto & t\bar{A}^p t^{-1}.
\end{array} \tag{3.3}$$

Let $\{\alpha_i = \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)\}_{i=0}^{p-1}$ be the coset representatives of the group $\Phi_r^s \cap \Gamma^0(p)$ in Φ_r^s. If we use these representatives to define the map in (3.3), then the transfer map looks like $\bar{A} \mapsto \bar{A}^p$. By Remark 3.1, $tA^p t^{-1}$ and A^p represent the same coset mod $\Gamma_1(Np^r)^{\mathrm{ab}}$ and hence we are done. □

We would like to define an action of Γ on $\Phi_r^{s\,\mathrm{ab}}$ and call it the nebentypus action. This can be done as follows: For $r \geq 2$, if $\bar{d} \in \Gamma/\Gamma_r$, then choose an element $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$ such that $p^{r+1} \mid c$ and $p \mid b$, i.e., $\alpha \in \Phi_{r+1}^2 \cap \Gamma^0(p)$. Such an α exists, because

$$\Phi_{r+1}^2 \cap \Gamma^0(p) \twoheadrightarrow \Gamma/\Gamma_{r+1} \twoheadrightarrow \Gamma/\Gamma_r.$$

The nebentypus action of d on $\Phi_r^{s\,\mathrm{ab}}$ is given by conjugation by α. This action is well-defined because if α_1 and α_2 denote two lifts of $\bar{d}$, then $\alpha_1^{-1}\alpha_2 \in \Gamma_1(Np^{r+1}) \cap \Gamma^0(p) \subseteq \Phi_r^s$ and hence for any element $x \in \Phi_r^s$, $\alpha_1^{-1}\alpha_2 x \alpha_2^{-1}\alpha_1 = x$ in $\Phi_r^{s\,\mathrm{ab}}$. Now we shall show that the actions of U and Γ commutes.

Lemma 3.5. *If $r \geq s > 1$, the actions of U and Γ commutes on $\Phi_r^{s\,\mathrm{ab}}$.*

Proof. Though the proof of this lemma is similar to the proof of [1, Lem. 3.5.], here we make some remarks in between, hence we briefly recall its proof. It is easy to see that $\alpha(\Phi_r^s \cap \Gamma^0(p))\alpha^{-1} = \Phi_r^s \cap \Gamma^0(p)$ for any $\alpha \in \Phi_{r+1}^1 \cap \Gamma^0(p)$, since $\alpha\Phi_r^s\alpha^{-1} \subseteq \Phi_r^s$. Look at the following commutative diagram

$$\begin{tikzcd}
\Phi_r^{s\,\mathrm{ab}} \arrow[r, "\alpha-\alpha^{-1}"] \arrow[d, "V"] & \Phi_r^{s\,\mathrm{ab}} \arrow[d, "V"] \\
(\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[r, "\alpha-\alpha^{-1}"] \arrow[d, "t-t^{-1}"] & (\alpha(\Phi_r^s \cap \Gamma^0(p))\alpha^{-1})^{\mathrm{ab}} = (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[d, "\alpha t\alpha^{-1}(-)\alpha t^{-1}\alpha^{-1}"] \\
\Phi_{r+1}^{s\,\mathrm{ab}} \arrow[r, "\alpha-\alpha^{-1}"] \arrow[d] & (\alpha\Phi_{r+1}^s\alpha^{-1})^{\mathrm{ab}} = \Phi_{r+1}^{s\,\mathrm{ab}} \arrow[d] \\
\Phi_r^{s\,\mathrm{ab}} \arrow[r, "\alpha-\alpha^{-1}"] & (\alpha\Phi_r^s\alpha^{-1})^{\mathrm{ab}} = \Phi_r^{s\,\mathrm{ab}}.
\end{tikzcd}$$

The top square in the diagram above commutes because if $\{\gamma_1, \ldots, \gamma_q\}$ form a set coset representatives for the group $\Phi_r^s \cap \Gamma^0(p)$ in Φ_r^s, so is the set $\{\alpha\gamma_1\alpha^{-1}, \ldots, \alpha\gamma_q\alpha^{-1}\}$. Observe that, this diagram commutes even if $\alpha \in \Phi_r^1 \cap \Gamma^0(p)$. The last square commutes by the functoriality of the transfer map.

We now prove the commutativity of the middle square, i.e., the map

$$\alpha t\alpha^{-1}(-)\alpha t^{-1}\alpha^{-1} : (\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \to \Phi_{r+1}^{s\,\mathrm{ab}} \tag{3.4}$$

is $t - t^{-1}$. If $g \in \Phi_r^s \cap \Gamma^0(p)$, then

$$\alpha t\alpha^{-1} g \alpha t^{-1}\alpha^{-1} = (\alpha t\alpha^{-1}t^{-1}) t g t^{-1} (\alpha t\alpha^{-1}t^{-1})^{-1}.$$

Since $\alpha t\alpha^{-1}t^{-1} \in \Gamma_1(Np^{r+1})$ for $\alpha \in \Phi_{r+1}^1 \cap \Gamma^0(p)$, we see that the conjugation by $\alpha t\alpha^{-1}t^{-1}$ induces identity on $\Phi_{r+1}^{s\,\mathrm{ab}}$ (because elements of Φ_{r+1}^s do commute in $\Phi_{r+1}^{s\,\mathrm{ab}}$).

In the above diagram composition of the vertical morphisms on either side are the operator U and it commutes with the automorphism of Φ_r^s induced by conjugation by α, but we know Γ acts on Φ_r^s by conjugation by such elements α. $\square$

Observe that the inclusion $\Gamma_1(Np^r) \subseteq \Phi_r^s$ gives rise to the another transfer map

$$\Phi_r^{s\,\mathrm{ab}} \xrightarrow{V} \Gamma_1(Np^r)^{\mathrm{ab}}$$

Lemma 3.6. *The transfer morphism $V : \Phi_r^{s\,\mathrm{ab}} \to \Gamma_1(Np^r)^{\mathrm{ab}}$ commutes with the action of U on its source and target.*

Proof. It suffices to prove that the following diagram (in which V denotes the transfer maps between various abelianizations) commutes:

$$\begin{tikzcd}
\Phi_r^{s\,\mathrm{ab}} \arrow[r, "V"] \arrow[d, "V"] & \Phi_r^{r\,\mathrm{ab}} \arrow[d, "V"] \\
(\Phi_r^s \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[r, "V"] \arrow[d, "t-t^{-1}"] & (\Phi_r^r \cap \Gamma^0(p))^{\mathrm{ab}} \arrow[d, "t-t^{-1}"] \\
\Phi_{r+1}^{s\,\mathrm{ab}} \arrow[r, "V"] & \Phi_{r+1}^{r\,\mathrm{ab}}.
\end{tikzcd}$$

The top square in the diagram above commutes because of functoriality of the transfer map. The commutativity of the bottom square follows by the following calculation.

If $\sigma_d = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where d runs through coset representatives of Γ_r in Γ_s, forms a set of coset representatives for the group $\Gamma_1(Np^r) \cap \Gamma^0(p)$ in $\Phi_r^s \cap \Gamma^0(p)$, then so are $t\sigma_d t^{-1} = \left(\begin{smallmatrix} a & b/p \\ cp & d \end{smallmatrix}\right)$ for the group $\Gamma_1(Np^r)$ in Φ_r^s (by Remark 2.3). ☐

In this section, we have defined the U-operators for the congruence subgroups $\{\Phi_r^s\}$ and proved that morphisms between these congruence subgroups respects the action of U and this action commutes with the action of Γ.

4. Ordinary parts

Let A be a commutative finite $\mathbb{Z}_p$-algebra and U be a non-zero element of A. It well-known that A factors as a product of local rings. Let A^{ord} denote the product of all those local rings of A in which the projection of U is a unit. This is a flat A-algebra.

Let M be any module in the abelian category of $\mathbb{Z}_p[X]$-modules which are finitely generated as $\mathbb{Z}_p$-modules. In this case, we take A to be the image of $\mathbb{Z}_p[X]$ in $\mathrm{End}_{\mathbb{Z}_p}(M)$, which is a finite $\mathbb{Z}_p$-algebra, and U to be the image of X. We define

$$M^{\mathrm{ord}} := M \otimes_A A^{\mathrm{ord}}$$

and call this the ordinary part of M. Observe that taking ordinary parts is an exact functor on our abelian category.

If we consider X to be the U-operator corresponding to the prime p, we may consider the ordinary part of the $\mathbb{Z}_p$-homology of the curve Y_r, i.e., the module $(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$, which is a Γ-module by Lemma 3.5.

We have the following theorem for the prime $p = 2$, which is similar to Theorem 3.1 in [3] for $p \geq 5$ and for the congruence subgroups $\Gamma_1(Np^r)$ for $r \geq 1$.

Theorem 4.1. *If $r \geq s > 1$, then the morphism of abelian groups*

$$(\Gamma_1(Np^r) \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \to (\Gamma_1(Np^s) \otimes \mathbb{Z}_p)^{\mathrm{ord}}$$

is an isomorphism.

Proof. We shall show that

$$(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \xrightarrow{\sim} (\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \xrightarrow{\sim} (\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}. \tag{4.1}$$

If $\pi : \Phi_r^{s\,\mathrm{ab}} \to \Phi_{r-1}^{s\,\mathrm{ab}}$ is the morphism induced by the inclusion $\Phi_r^s \subset \Phi_{r-1}^s$, then

$$U' \circ \pi = U \in \mathrm{End}(\Phi_r^{s\,\mathrm{ab}}),\ \pi \circ U' = U \in \mathrm{End}(\Phi_{r-1}^{s\,\mathrm{ab}}).$$

By Lemma 3.2, we have the following diagram

$$\begin{tikzcd}
\Phi_{r-1}^{s\,\mathrm{ab}} \arrow[r, "\pi"] \arrow[d, "U'"] \arrow[dr, "U"] & \Phi_{r-2}^{s\,\mathrm{ab}} \arrow[d, "U'"] \\
\Phi_{r}^{s\,\mathrm{ab}} \arrow[r, "\pi"] \arrow[d, "U'"] \arrow[dr, "U"] & \Phi_{r-1}^{s\,\mathrm{ab}} \arrow[d, "U'"] \\
\Phi_{r+1}^{s\,\mathrm{ab}} \arrow[r, "\pi"] & \Phi_{r}^{s\,\mathrm{ab}}.
\end{tikzcd}$$

The existence of U' implies that upon tensoring over $\mathbb{Z}_p$ and taking the ordinary parts π induces an isomorphism (and $U^{-1} \circ U'$ provides an inverse to π)

$$(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} = (\Phi_{r-1}^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

By induction on r, we obtain the second isomorphism in (4.1), i.e.,

$$(\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} = (\Phi_s^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} = (\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}.$$

To prove the first isomorphism consider the short exact sequence

$$1 \to \Gamma_1(Np^r)^{\mathrm{ab}}/\mathfrak{a}_s \to \Phi_r^{s\,\mathrm{ab}} \to (\Gamma_s/\Gamma_r) \to 1.$$

By tensoring this sequence with $\mathbb{Z}_p$ and then taking the ordinary parts to obtain

$$1 \to (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s \to (\Phi_r^{s\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} \to (\Gamma_s/\Gamma_r)^{\mathrm{ord}} \to 1,$$

because $\mathbb{Z}_p$ is flat as a $\mathbb{Z}$-module and ordinary parts preserves exactness. By Lemma 3.4, the operator U acts on Γ_s/Γ_r as multiplication by p and so is a nilpotent operator, as Γ_s/Γ_r is a p-torsion group. Thus $(\Gamma_s/\Gamma_r)^{\mathrm{ord}} = 0$, and hence the Theorem follows. □

5. Iwasawa modules

We have the following inverse system indexed by natural numbers $r \geq 2$,

$$\cdots \to \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p \to \cdots \to \Gamma_1(Np^2)^{\mathrm{ab}} \otimes \mathbb{Z}_p.$$

Define the Iwasawa module by

$$\mathbf{W} := \varprojlim_{r \geq 2} \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p.$$

The profinite group Γ acts on the $\mathbb{Z}_p$-module $\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p$ through its finite quotient Γ/Γ_r. Thus the Iwasawa module $\mathbf{W}$ becomes a module over the completed group algebra

$$\Lambda := \mathbb{Z}_p[[\Gamma]] = \varprojlim_{r \geq 2} \mathbb{Z}_p[\Gamma/\Gamma_r].$$

Though the Iwasawa module $\mathbf{W}$ is difficult to understand, by Theorem 4.1, we can understand the ordinary part of $\mathbf{W}$ very well. To make the statement clear, let us slightly abstract the situation.

Let $\{M_r\}_{r\geq 2}$ be a system of Λ-modules. Further, assume that each M_r is pointwise fixed by Γ_r and hence a module over $\Lambda/\mathfrak{a}_r\Lambda = \mathbb{Z}_p[\Gamma/\Gamma_r]$. For each $r \geq s \geq 2$, we have a map $M_r \to M_s$ such that it factors via

$$M_r/\mathfrak{a}_s M_r \to M_s.$$

Define $W := \varprojlim_{r\geq 2} M_r$. We have a collection of maps $W \to M_r$ for each $r \geq 2$ and they factor as

$$W/\mathfrak{a}_r W \to M_r.$$

Proposition 5.1. *Assume that each M_r is p-adically complete and for each $r \geq s \geq 2$, $M_r/\mathfrak{a}_s M_r \to M_s$ is an isomorphism. Then $W/\mathfrak{a}_s W \twoheadrightarrow M_s$ is an isomorphism.*

Proof. For $r \geq s \geq 2$, the maps $M_r \to M_s$ are surjective, and hence the canonical map from $W \to M_s$ is also surjective. We shall show that the kernel is $\mathfrak{a}_s W$.

Since each M_r is p-adically complete and is point-wise fixed by Γ_r, we have $M_r = \varprojlim_i M_r/\mathfrak{n}^i M_r$, where $\Gamma_r = \overline{\langle \gamma_r \rangle}$ and $\mathfrak{n} = (\gamma_r - 1, p)$, i.e., each M_r is $\mathfrak{n}$-adically complete.

By induction on i, we get that $\gamma_s^{p^i} - 1/\gamma_s - 1 \in (\gamma_s - 1, p)^i$. In particular, we have $\gamma_2^{p^{r-2}} - 1/\gamma_2 - 1 \in \mathfrak{m} = (\gamma_2 - 1, p)$. Hence, $\mathfrak{m}^{p^{r-2}} \subseteq ((\gamma_2 - 1)^{p^{r-2}}, p) \subseteq \mathfrak{n} \subseteq \mathfrak{m} = (\mathfrak{a}_2, p)$. As a result, we see that each M_r is $\mathfrak{m}$-adically complete, since they are $\mathfrak{n}$-adically complete. Once we have that each M_r is $\mathfrak{m}$-adically complete, then proving the injectivity of the above map is quite similar to the proof of [1, Prop. 5.1]. □

The following Theorem is an immediate consequence of the Proposition above.

Theorem 5.2. *For any $r \geq 2$, we have*

$$\mathbf{W}^{\mathrm{ord}}/\mathfrak{a}_r \mathbf{W}^{\mathrm{ord}} \cong (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$$

is the Γ_r-co-invariants of $\mathbf{W}^{\mathrm{ord}}$.

Proof. This follows from Proposition 5.1 together with Theorem 4.1 □

The above Theorem is a key ingredient for the proof of the Theorem 5.3. The Λ-module $\mathbf{W}^{\mathrm{ord}}$ is a compact Λ-module (under the projective limit of the p-adic topologies on each module $\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p$, which are free of finite rank over $\mathbb{Z}_p$, and also since $\mathbf{W}^{\mathrm{ord}}$ is a direct factor of $\mathbf{W}$).

Furthermore, Theorem 5.2 implies that the projective limit topology on $\mathbf{W}^{\mathrm{ord}}$ coincides with its $\mathfrak{m}$-adic topology (where $\mathfrak{m} = (\mathfrak{a}_2, p) \subset \Lambda$ denotes the maximal ideal of Λ), because the kernels of the projections $\Lambda \to \mathbb{Z}_p/p^r\mathbb{Z}_p[\Gamma/\Gamma_r]$ are co-final with the sequence of ideals $\mathfrak{m}^r$ in Λ. Thus $\mathbf{W}^{\mathrm{ord}}$ is a Λ-module, compact in its $\mathfrak{m}$-adic topology such that

$$\mathbf{W}^{\mathrm{ord}}/\mathfrak{m} = \mathbf{W}^{\mathrm{ord}}/(\mathfrak{a}_2, p) = (\Gamma_1(Nq)^{\mathrm{ab}} \otimes \mathbb{Z}_p/p)^{\mathrm{ord}}$$

is a finite dimensional $\mathbb{Z}_p/p\mathbb{Z}_p$-module, of dimension d (say). By Nakayama's lemma, we have that $\mathbf{W}^{\mathrm{ord}}$ is a finitely generated Λ-module with a minimal generating set has cardinality d. We have the following theorem for the prime $p = 2$, which is similar to the main theorem in [3] for $p \geq 5$.

Theorem 5.3 (Main Result)**.** *The module $\mathbf{W}^{\mathrm{ord}}$ is free of finite rank over Λ, and its Λ-rank is equal to d.*

As a corollary, we see that, for $r \geq 2$, the $\mathbb{Z}_p$-rank of the free $\mathbb{Z}_p$-module $(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$ is d. In particular, these $\mathbb{Z}_p$-ranks are independent of p^r in the level. Using this result, we have proved control theorems for ordinary 2-adic families of modular forms, see [2]. The classical versions of this theorem for $p = 2, 3$ do not seem to be explicitly available in the literature, though an adèlic version of it can be found in [4].

6. Reflexivity results

To prove Theorem 5.3, it enough to show that $\mathbf{W}^{\mathrm{ord}}$ is a reflexive Λ-module [5]. We show this by considering the duality theory of the modules $(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}$ and showing that they are reflexive as $\mathbb{Z}_p[\Gamma/\Gamma_r]$-modules. Now, we briefly recall the notion of reflexivity, and the necessary results. For more details, see [1, §6].

Suppose that R is a commutative ring, G is a finite group, and M is a left $R[G]$-module. Let N be any R-module. Then $\mathrm{Hom}_R(M, N)$ is a right $R[G]$-module, via

$$(f * g)(x) := f(g^{-1}x).$$

Since the ring $R[G]$ is naturally a bi-module over itself, via the ring multiplication, $R[G] \otimes_R N$ is an $R[G]$-bi-module, making $\mathrm{Hom}_{R[G]}(M, R[G] \otimes_R N)$ a right $R[G]$-module.

Lemma 6.1 ([1])**.** *There is a canonical isomorphism of right $R[G]$-modules*

$$\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{R[G]}(M, R[G] \otimes_R N).$$

In particular, when $N = R$, we see that M^* and $\mathrm{Hom}_{R[G]}(M, R[G])$ are canonically isomorphic as right $R[G]$-modules, where $M^* := \mathrm{Hom}_R(M, R)$, the R-dual of M. The analogue of the above lemma for right $R[G]$-modules is also true. Hence,

$$\mathrm{Hom}_R(M^*, R) = \mathrm{Hom}_{R[G]}(M^*, R[G]).$$

are canonically isomorphic as left $R[G]$-modules.

By definition of M^*, there is a natural morphism of R-modules $M \to \mathrm{Hom}_R(M^*, R)$, which is also a morphism of left $R[G]$-modules. If this natural morphism of R-modules is an isomorphism, then we say that M is a reflexive R-module. Thus we have proved:

Lemma 6.2. *If M is a left $R[G]$-module which is reflexive as an R-module, then M is reflexive as an $R[G]$-module.*

The crux of this Lemma is that to check the reflexivity of $R[G]$-module M over $R[G]$, it is enough to check it over R. Now we need to understand how to use the reflexivity results for modules over $\mathbb{Z}_p[\Gamma/\Gamma_r]$ to show the reflexivity of $\mathbf{W}^{\mathrm{ord}}$ as a Λ-module.

7. Proof of Theorem 5.3

For $r \geq 2$, and $N \in \mathbb{N}$ such that $(p, N) = 1$. We define the cohomology of Y_r as

$$\mathrm{H}^1(Y_r, \mathbb{Z}_p) := \mathrm{Hom}_{\mathbb{Z}}(\Gamma_1(Np^r)^{\mathrm{ab}}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p).$$

The ring Λ acts on $\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p$ through its quotient $\Lambda_r := \Lambda/\mathfrak{a}_r = \mathbb{Z}_p[\Gamma/\Gamma_r]$. More generally, if $r \geq s > 1$ then the ring Λ_s is equal to $\Lambda_r/\mathfrak{a}_s$, hence $\Lambda_r \twoheadrightarrow \Lambda_s$. Thus we get the following sequence of morphisms of Λ_r-modules

$$\begin{aligned}&\mathrm{Hom}_{\Lambda_r}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \Lambda_r) \to \mathrm{Hom}_{\Lambda_r}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \Lambda_r)/\mathfrak{a}_s \\ &\to \mathrm{Hom}_{\Lambda_r}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \Lambda_s) = \mathrm{Hom}_{\Lambda_s}(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p/\mathfrak{a}_s, \Lambda_s).\end{aligned}$$

If M is any $\mathbb{Z}_p[U]$-module, which is finitely generated as a $\mathbb{Z}_p$-module, then so is the $\mathbb{Z}_p$-dual $M^* := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$. Here M^* is a $\mathbb{Z}_p[U]$-module via the dual action of U. Clearly $(M^*)^{\mathrm{ord}} = (M^{\mathrm{ord}})^*$, i.e., taking ordinary parts commutes with duals. Thus

we may take ordinary parts of the above diagram of homomorphisms to obtain a diagram

$$\begin{aligned}\mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r) &\longrightarrow \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r)/\mathfrak{a}_s\\ &\longrightarrow \mathrm{Hom}_{\Lambda_s}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s,\Lambda_s).\end{aligned}$$

By Theorem 4.1, we have

$$\begin{aligned}\mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r) &\longrightarrow \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r)/\mathfrak{a}_s\\ &\longrightarrow \mathrm{Hom}_{\Lambda_s}((\Gamma_1(Np^s)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_s).\end{aligned}$$

Lemma 7.1. *The morphism*

$$\mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r)/\mathfrak{a}_s \to \mathrm{Hom}_{\Lambda_s}((\Gamma_1(Np^s)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_s)$$

is an isomorphism.

Proof. By Lemma 3.6, we may restrict V to the ordinary parts to obtain a morphism

$$(\Phi_r^{s\,\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}} \xrightarrow{V} (\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}}.$$

Look at the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\mathbb{Z}_p) \arrow[r, "\sim"] \arrow[d, "V^*"] & \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r) \arrow[d] \\
\mathrm{Hom}_{\mathbb{Z}_p}((\Phi_r^{s\,\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\mathbb{Z}_p) \arrow[dd, "\wr"] & \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_r)/\mathfrak{a}_s \arrow[d] \\
 & \mathrm{Hom}_{\Lambda_s}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}}/\mathfrak{a}_s,\Lambda_s) \arrow[d, "\wr"] \\
\mathrm{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^s)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\mathbb{Z}_p) \arrow[r, "\sim"] & \mathrm{Hom}_{\Lambda_s}((\Gamma_1(Np^s)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\Lambda_s)
\end{tikzcd}$$

in which the two horizontal isomorphisms are those provided by Lemma 6.2, because $\Lambda_r = \mathbb{Z}_p[\Gamma/\Gamma_r]$. The first vertical map V^* is the dual morphism of V and the two vertical isomorphisms are a part of Theorem 4.1 and its proof.

Now to prove the Lemma, it suffices to prove that

$$\mathrm{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\mathbb{Z}_p) \xrightarrow{V^*} \mathrm{Hom}_{\mathbb{Z}_p}((\Phi_r^{s\,\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\mathbb{Z}_p) \tag{7.1}$$

is surjective and $\mathrm{kernel}(V^*) = \mathfrak{a}_s\mathrm{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p)^{\mathrm{ord}},\mathbb{Z}_p))$.

Since V commutes with U and taking ordinary parts commutes with taking $\mathbb{Z}_p$-duals, the morphism in (7.1) is the ordinary part of the morphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{V^*} \mathrm{Hom}_{\mathbb{Z}_p}(\Phi_r^{s\,\mathrm{ab}}\otimes\mathbb{Z}_p,\mathbb{Z}_p). \tag{7.2}$$

Now, it suffices to show that the morphism V^* in (7.2) is surjective with kernel equal to $\mathfrak{a}_s\mathrm{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{\mathrm{ab}}\otimes\mathbb{Z}_p),\mathbb{Z}_p)$, since taking ordinary parts is also exact and commutes with the action of Γ. But, this claim was proved in [1, §8] for torsion-free groups H and G such that $H\subseteq G$, instead of $\Gamma_1(Np^r)\subseteq\Phi_r^s$. Observe that, when $p=2$ and $r\geq s\geq 2$, the groups $\Gamma_1(Nq)$ and Φ_r^s are torsion-free, since $\Gamma_1(M)$ is torsion free for all $M\geq 3$. □

We now have all the information needed to prove Theorem 5.3. Consider the chain of Λ-modules

$$\cdots \longrightarrow \mathrm{Hom}_{\Lambda_r}((\Phi_r^{r\,\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}((\Gamma_1(Nq)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p).$$

Lemma 7.2. *There is a canonical isomorphism*

$$\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}}, \Lambda) = \varprojlim_r \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r).$$

Proof. We have the following canonical isomorphisms

$$\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}}, \Lambda) = \varprojlim_r \mathrm{Hom}_{\Lambda_r}(\mathbf{W}^{\mathrm{ord}}/\mathfrak{a}_r, \Lambda_r) = \varprojlim_r \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r),$$

where the last isomorphism follows from the Theorem 5.2. ☐

Proposition 7.3. *For $r > 1$, there is a canonical isomorphism*

$$\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}}, \Lambda)/\mathfrak{a}_r = \mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r).$$

Proof. The claim follows from Lemma 7.1, Lemma 7.2, and Lemma 5.1. ☐

Theorem 7.4. *The module $\mathbf{W}^{ord}$ is Λ-free.*

Proof. Since any finitely generated reflexive Λ-module is free, it suffices to show that $\mathbf{W}^{\mathrm{ord}}$ is a reflexive Λ-module. By Proposition 7.3 and Lemma 6.2, we have:

$$\begin{aligned}
\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\mathbf{W}^{\mathrm{ord}}, \Lambda), \Lambda) &= \varprojlim_r \mathrm{Hom}_\Lambda(\mathrm{Hom}_{\Lambda_r}(\mathbf{W}^{\mathrm{ord}}, \Lambda)/\mathfrak{a}_r, \Lambda_r) \\
&= \varprojlim_r \mathrm{Hom}_{\Lambda_r}(\mathrm{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}}, \Lambda_r), \Lambda_r) \\
&= \varprojlim_r \, (\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbb{Z}_p)^{\mathrm{ord}} = \mathbf{W}^{\mathrm{ord}}.
\end{aligned}$$

☐

8. Acknowledgements

The author thanks Prof. M. Emerton for his encouragement to work out the details in [1] for the prime $p = 2$. These results turned out be quite useful in [2].

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